43 marks

PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. Independent Public School	Year 12 Specialist TEST 5 20 Aug 2018 TIME: 50 minutes working No notes! Classpads allowed. 43 Marks 9 Questions

Name:______________________ Teacher: ______________________

NOTE: -1 max for no constant

Q1 (2 & 2 = 4 marks)
Determine the general solution for the following.

a) $5y\dfrac{dy}{dx} = 1 - 7x$

$$\int 5y\,dy = \int (1 - 7x)\,dx$$

$$\frac{5y^2}{2} = x - \frac{7x^2}{2} + C$$

✓ separation of variables
✓ expression with constant

b) $\dfrac{dy}{dx} = \dfrac{x(1-3x)}{\sin y}$

$$\int \sin y\,dy = \int (x - 3x^2)\,dx$$

$$-\cos y = \frac{x^2}{2} - x^3 + C$$

✓ separation of variables
✓ expression with a constant

Q2 (4 marks)

A hot item, initially at $315^\circ C$, is placed in a room with temperature $21^\circ C$ and left to cool, the temperature $T^\circ C$ of the item t minutes later is given by the differential equation

$$\frac{dT}{dt} = -3(T - 21)$$

Determine how long it will take for the temperature of the item to cool to $100^\circ C$.

$$\int \frac{dT}{(T-21)} = \int -3\,dt$$

$$\ln(T-21) = -3t + C$$

$$T - 21 = Pe^{-3t}$$

$$T = Pe^{-3t} + 21$$

$t=0$

$$315 = P + 21$$

$$P = 294$$

$$100 = 294e^{-3t} + 21$$

$$t = 0.438 \text{ mins}$$

$$\approx 26 \text{ seconds}$$

✓ separation of variables
✓ integrates giving a constant
✓ solves for constant AND t in mins for $100^\circ C$
✓ determines time to nearest second.

Q3 (2, 4 & 3 = 9 marks)
The logistical growth model is given by the following differential equation.

$\frac{dy}{dx} = ay - by^2$ where $a \,\&\, b$ are positive constants and $y > 0$

a) State the y value where the gradient will be zero and hence give the limiting value of y.

$y(a-by)=0 \qquad a=by \qquad y=\frac{a}{b}$

✓ equates $\frac{dy}{dx}$ to zero
✓ states limiting value

b) Using separation of variables and partial fractions, derive the logistical formula

$y = \frac{a}{b+Ce^{-ax}}$ where C is a constant. Show all steps without the use of a classpad.

$$\frac{dy}{dx} = y(a-by)$$

$$\int \frac{dy}{y(a-by)} = \int dx$$

$$\frac{1}{y(a-by)} = \frac{A}{y} + \frac{B}{a-by}$$

$$1 = A(a-by) + By$$

$y=0$

$1 = Aa \qquad A=\frac{1}{a}$

$y=\frac{a}{b}$

$1 = 0 + B\frac{a}{b} \qquad B=\frac{b}{a}$

$$\frac{1}{a}\int\left(\frac{1}{y}+\frac{b}{a-by}\right)dy = x + C$$

$$\ln y - \ln(a-by) = ax + C$$

Note as $y < \frac{a}{b}$
$a - by > 0$

$$\ln\frac{y}{a-by} = ax + C$$

$$\frac{y}{a-by} = Ce^{ax}$$

$$\frac{a-by}{y} = Ce^{-ax}$$

$$a-by = Cye^{-ax}$$

$$a = y(b+Ce^{-ax})$$

$$y = \frac{a}{b+Ce^{-ax}}$$

✓ separation of variables
✓ uses partial fractions
✓ derives expression with natural logs AND stating $a-by>0$ OR use absolute value.
✓ derives final expression.

Q3 continued

c) Given that the Population P of a group of Kangaroos at t years (initially 285 kangaroos) can be modelled by the logistical growth model $\frac{dP}{dt} = \frac{1}{4}P - \frac{1}{13780}P^2$, determine the time taken for the population to reach 2000 kangaroos. Use your result from (b)

$a = \frac{1}{4}$ $b = \frac{1}{13780}$

$P = \dfrac{\frac{1}{4}}{\frac{1}{13780} + Ce^{-\frac{1}{4}t}}$

✓ uses a + b values into formula

$285 = \dfrac{\frac{1}{4}}{\frac{1}{13780} + C}$

✓ Solves for constant C (approx)

$C = \frac{158}{196365}$

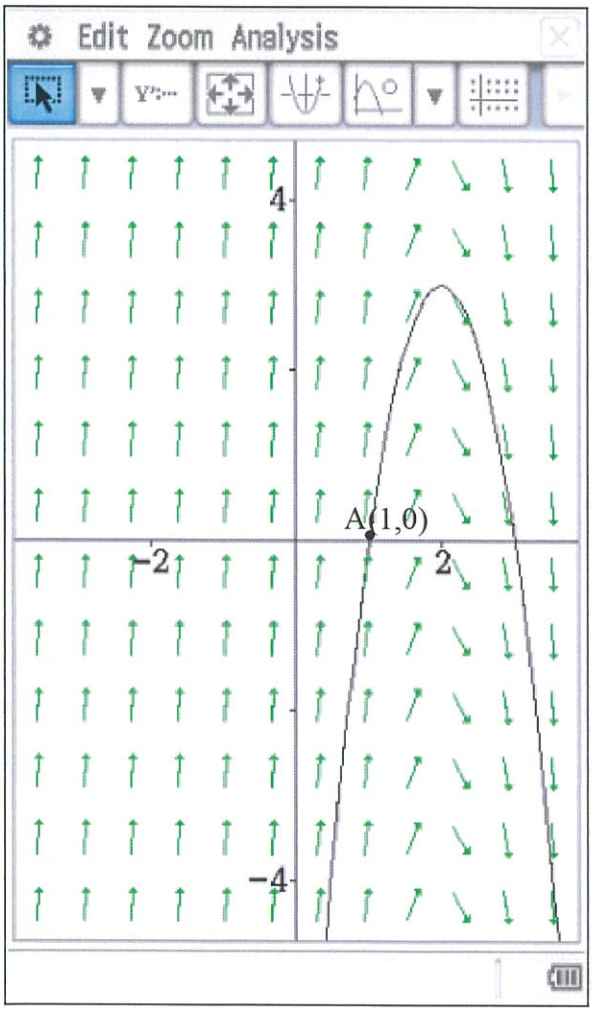
$2000 = \dfrac{\frac{1}{4}}{\frac{1}{13780} + \frac{158}{196365}e^{-\frac{1}{4}t}}$

$t = 10.92$ yrs

✓ solves for time (approx)

Q4 (4 marks)

A slope field is plotted below showing a particular line of force through point A(1,0). At point A the slope field is 6.



Given that the slopes are horizontal at x=2 and that the lines of force are parabolic.
Determine the slope field in terms of x.

$y = a(x-2)^2 + C$

$0 = a + C \quad \therefore C = -a$

$y = a(x-2)^2 - a$

$\frac{dy}{dx} = 2a(x-2)$

$6 = -2a \quad \therefore a = -3$

$y = -3(x-2)^2 + 3$

$\frac{dy}{dx} = -6(x-2)$

✓ uses $y = a(x-2)^2 + C$ or two constants

✓ solves for C in terms of a

✓ uses $\frac{dy}{dx}$ at pt A to solve for a

✓ gives both y equation AND slope field function.

Q5 (4 marks)
An object is moving in a straight line so that its speed, v metres per second, at displacement x metres form the origin at time t seconds can be described by the following acceleration. The speed is zero when x=1 metre from the origin.

$$\frac{dv}{dt} = x\left(5+3x^2\right)^5$$

Determine the speed when $x = 5$ metres.

$$v\frac{dv}{dx} = x(5+3x^2)^5$$

$$\int v\,dv = \int x(5+3x^2)^5\,dx$$

$$\frac{1}{2}v^2 = A(5+3x^2)^6 + C$$

(diff eqn $6A(+6x)(5+3x^2)^5$

$1 = +36A$

$A = +\frac{1}{36}$

$$\frac{1}{2}v^2 = +\frac{1}{36}(5+3x^2)^6 + C$$

$0 = +\frac{8^6}{36} + C \quad \therefore C = -\frac{8^6}{36}$

$$\frac{1}{2}v^2 = +\frac{(5+3x^2)^6}{36} - \frac{8^6}{36}$$

~~$V \approx$ [illegible] m/s~~

$V \approx 120679.5$ m/s
approx

Q6 (4 marks)
A particle is undergoing Simple Harmonic Motion and can be described by $\ddot{x} = -36x$.
Determine what percentage of the time that the particle is less than three quarters of the maximum distance from the origin.

$n^2 = 36$

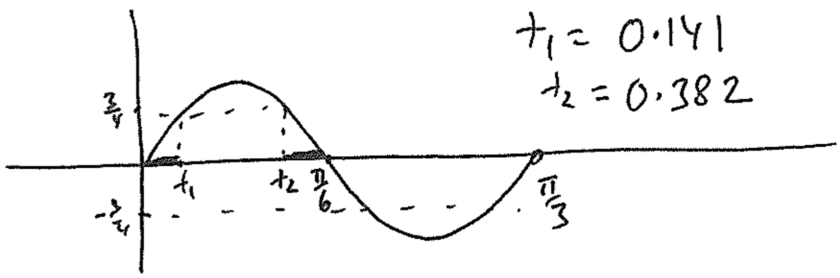
$n = 6$

$T = \frac{2\pi}{6} = \frac{\pi}{3}$

$x = A\sin(6t + \delta)$

$t_1 = 0.141$

$t_2 = 0.382$



✓ uses $n = 6$

✓ identifies period

✓ determines times that $\sin(\ ) = \frac{3}{4}$

✓ determines %

$$\% \approx \frac{0.141 + \left(\frac{\pi}{6} - 0.382\right)}{\frac{\pi}{6}}$$

$\approx 53.9\%$ (accept 54%)

Q7 (3 & 2 = 5 marks)
An object is undergoing SHM $\ddot{x} = -4x$ and is initially at rest with $x = 15$ units but with a positive initial acceleration.
Determine.

a) An expression for x in terms of time, t.

$x = -15\cos 2t$ or $15\cos(2t+\pi)$

✓ Amplitude
✓ n value
✓ correct phase constant

b) The distance travelled in the first 10 seconds.

$\dot{x} = 30\sin 2t$

✓ velocity
✓ integral with absolute value

$\int_0^{10} |30\sin 2t|\, dt = 188.87$ units

Q8 (3 & 3 =6 marks)
An object's displacement, x metres at time, t seconds is described by
$x = 7\cos(3t) - 5\sin(3t)$

a) Show that the motion is Simple Harmonic.

$\dot{x} = -21\sin 3t - 15\cos 3t$
$\ddot{x} = -63\cos 3t + 45\sin 3t$
$= -9(7\cos 3t - 5\sin 3t)$
$\ddot{x} = -9x$ hence SHM

✓ determines velocity
✓ determines acceleration
✓ shows that $\ddot{x} = -n^2 x$

b) Determine the Amplitude and the **exact** speed when $x = 4$ metres.

$A = \sqrt{7^2 + 5^2}$
$= \sqrt{74}$

$n = 3$

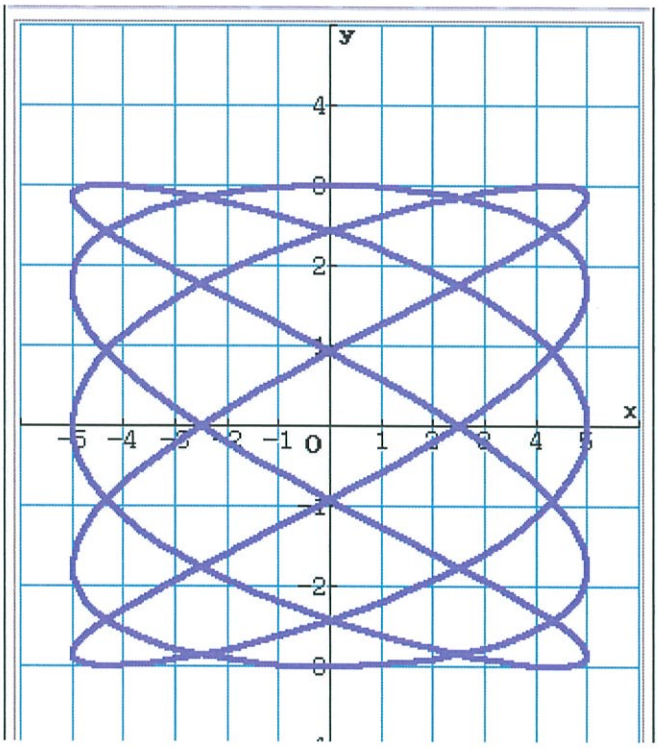
$v^2 = n^2(A^2 - x^2)$
$v^2 = 9(74 - x^2)$
$x = 4$
$v^2 = 9(74 - 4^2)$
$v = 3\sqrt{58}$ m/s or $(\sqrt{522})$

✓ determines Amplitude
✓ uses $v^2 = n^2(A^2 - x^2)$
✓ determines exact speed

Q9 (3 marks)

The Iron Man completes a race following a unique race track so that his position vector in metres at time t seconds is given by $r = \begin{pmatrix} 5\cos\frac{2\pi}{3}t \\ 3\sin\frac{2\pi}{5}t \end{pmatrix}$.metres

The motion is graphed as follows.



Determine the time taken to complete one circuit of the race track and the length of this cicuit.

x motion $T = 3$ secs

y motion $T = 5$ secs

LCM = 15 seconds

✓ states 15 seconds

$$\dot{r} = \begin{pmatrix} -\frac{10\pi}{3}\sin\frac{2\pi}{3}t \\ \frac{6\pi}{5}\cos\frac{2\pi}{5}t \end{pmatrix}$$

✓ shows velocity function

✓ states integral to find distance (No need to evaluate)

$$\text{distance} = \int_0^{15} \sqrt{\left(-\frac{10\pi}{3}\sin\frac{2\pi}{5}t\right)^2 + \left(\frac{6\pi}{5}\cos\frac{2\pi}{5}t\right)^2}\, dt = 110.0 \text{ metres}$$